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Remarks on the mass spectrum on non-critical coset models from Toda theories

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Abstract. We discuss how coset lattice models perturbed in the appropriate direction should still be obtained by projections of non-critical integrable vertex models. The latter are described in the continuum limit by Toda field theories. By studying the mass spectrum of these theories and the projection mechanism, we conjecture without explicit construction of the S matrix the mass spectrum of the simplest coset models. Among cases considered are the unitary series perturbed in Φ_{13} direction, the $\mathbb{Z}_n$ models, the tricritical Ising and tricritical three-state Potts models perturbed by the thermal operator. The latter exhibit an E_7 and E_6 structure respectively. Some numerical checks are presented.

1. Introduction

Possible applications of conformal theory [1] to the study of the scaling region in 2D statistical mechanics systems are under active consideration [2-10].

In a series of recent works, Zamolodchikov [5-7] has devised general techniques for off-critical directions which preserve integrability. The latter property is present in any conformal invariant theory, an infinite set of integrals of motion being obtained by considering composite fields made up of $T(z)$ ($\bar{T}(\bar{z})$). As shown by Zamolodchikov, some of these integrals of motion can actually survive if one perturbs the fixed point action $\mathscr{A}^*$ by some well chosen relevant operator $\mathscr{O}$ (of weights (h, h), $h \leqslant 1$) to obtain a massive field theory. We write

$$\mathscr{A} = \mathscr{A}^* + \lambda \int \mathscr{O}(z, \bar{z})\, \mathrm{d}^2 z \tag{1}$$

where λ is a coupling constant of dimensions $(1-h, 1-h)\lambda \sim \xi^{2(h-1)}$. As in [5-7] we mainly consider first order in perturbation theory, so no counterterm is implied in (1). For any correlator, one finds at first order

$$\langle \partial_{\bar{z}} T \ldots \rangle_{\mathscr{A}} = \pi\lambda(1-h)\partial_z \langle \mathscr{O} \ldots \rangle_{\mathscr{A}}$$

or

$$\partial_{\bar{z}} T = \pi\lambda(1-h)\partial_z \mathscr{O}. \tag{2}$$

Hence a grade 1 conserved quantity is obtained in any case

$$Q_1 = \oint T(z, \bar{z})\, \mathrm{d}z \qquad \partial_{\bar{z}} Q_1 = 0. \tag{3}$$

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Depending on the theory and the perturbation considered, conserved quantities Q_n at grade n may be built [5, 7]. This occurs if there are some operators ϕ_n (with dimensions $h=n+1$, $\bar{h}=0$) and $\psi_n(\phi_1=T, \psi_1=(1-h)\mathcal{O}$ in (3)) such that†

$$\partial_{\bar{z}}\phi_n = \pi\lambda\partial_z\psi_n \tag{4}$$

(and Q_n is contour integral of ϕ_n). Equation (4) is equivalent to saying that the residue of the simple pole in the short distance expansion of ϕ_n with $\mathcal{O}$ is a total derivative

$$\phi_n(z)\mathcal{O}(w, \bar{w}) = \ldots + \frac{1}{z-w}\partial_z\psi_n(z, \bar{z}) + \text{regular terms} \tag{5}$$

This characterisation will be useful in the following. A last form of conservation deduced from (5) is

$$\left[\oint \phi_n(z)\,\mathrm{d}z, \oint \mathcal{O}(w, \bar{w})\,\mathrm{d}w\right] = 0 \tag{6}$$

which states that Q_n commutes with the perturbed Hamiltonian.

If enough conserved quantities can be built, the massive theory (1) is still integrable. In such cases, inelastic scattering is forbidden, and the S matrix factorises in terms of two particle scattering amplitudes which must satisfy the Yang–Baxter equation. In the simplest examples, analyticity, symmetry arguments, and knowledge of grades n for conserved quantities allow a complete determination of S-matrix elements. Using this line of thought, Zamolodchikov proposed the S matrix for the three-state Potts model perturbed by the thermal operator [6] (hence preserving $\mathbb{Z}_3$ symmetry), as well as the S matrix for the Ising model in a magnetic field (the latter exhibits in particular a beautiful E_8 structure). Recently, the Lee–Yang singularity has also been studied [8]. The knowledge of the S matrix allows in principle the determination of several interesting quantities [11, 12]. Among these are the masses of the particles, which are given by the poles of matrix elements in the complex rapidity plane [11]. In statistical mechanics models, these masses can be studied by analytic [13] or numerical [9] diagonalisation of transfer matrices.

The program initiated in [5–8] could be applied to the study of many off-critical systems (for instance unitary $c \leqslant 1$ models perturbed in the Φ_{13} direction). One may however wish to avoid case-by-case calculation of conserved quantities and S matrix in favour of a more general approach. From this point of view the possible relation between integrable directions and coset construction [14] revealed in [7] is appealing.

Rather than the GKO approach, we use in the following the related Feigin–Fuchs-like [15, 16] construction of conformal field theories. Recall that for $\mathcal{G}_k^{(1)} \otimes \mathcal{G}_1^{(1)}/\mathcal{G}_{k+1}^{(1)}$ where $\mathcal{G}_k^{(1)}$ is an affine simply laced Lie algebra at level k, this first involves an r component (where r is the rank of $\mathcal{G}$) free bosonic field theory of central charge $c=r$. Minimal models of central charge

$$c = r\left[r - \frac{h^v(h^v+1)}{(k+h^v)(k+h^v+1)}\right] \tag{7}$$

(where h^v is the Coxeter number of $\mathcal{G}$) are then obtained by adding a charge at infinity and restricting the Hilbert space with screening operators [17–19]. On the lattice there exists in principle a similar construction. The role of the free field should be played

† Notice that if ϕ_n, ψ_n satisfy (4), the same is true for their z derivatives, leading to trivial conserved quantities. Such cases are always omitted in the countings.

by a vertex model whose degrees of freedom are the $r+1$ vectors $\boldsymbol{\mu}=\Lambda_{\mu+1}-\Lambda_{\mu}$, $\mu=0,\ldots,r$, where the Λ_{μ} are the fundamental weights of $\mathscr{G}$. The charge at infinity should translate into boundary conditions [21], and the space of states be restricted [22] using the quantum group symmetry [23]. This scheme is completely worked out for A_1 [23], and partly for A_n [24]. Fewer progress have been made in the D or E cases, probably for technical reasons.

We explore in this paper the possibility of generalising the above structure to non-critical models. As a first step, we concentrate on the determination of mass spectra. We discuss how coset models perturbed in the appropriate direction (generalising Φ_{13} for A_1) should still be described by projections of non-critical vertex models (eight-vertex model [25] for A_1) with peculiar boundary conditions. The continuum limit of these vertex models turns out to be a Toda field theory [26] based on $\mathscr{G}^{(1)}$ (sine-Gordon for A_1), which is the natural non-critical extension of the r component free field of Feigin–Fuchs-like constructions. The mass spectrum of coset models is argued to be the same (up to degeneracies) than that of the Toda theory, which we relate to that of the nonlinear sigma model [27]. We discuss how this spectrum may be projected for minimal models, and finally get predictions for all $\mathscr{G}_k^{(1)}\otimes\mathscr{G}_1^{(1)}/\mathscr{G}_{k+1}^{(1)}$ theories perturbed in the $h=(k+1)/(k+h^{\mathrm{v}}+1)$ direction. In particular, in most of the spontaneously broken symmetry phases, the mass spectrum turns out to be continuous. Numerical checks are finally presented.

During this work, we received [10] by T Eguchi and S K Yang where some related ideas are discussed.

2. $c=1$ models

As a first exercise, we discuss some $c=1$ models. The simplest example is provided by the six-vertex (XXZ) model [25], which has a critical line parametrised by $\Delta=-\cos\gamma$, $\gamma\in[0,\pi]$, and renormalises onto a free bosonic theory with action

$$\mathscr{A}^*=\frac{g}{\pi}\int\partial_z\varphi\partial_{\bar z}\varphi\,\mathrm{d}^2z \tag{8}$$

where $g=1-\gamma/\pi$. At $\Delta=-1$, an infinite-order phase transition occurs, for $\Delta\geqslant-1$, there is antiferroelectric order. The phase $\Delta\geqslant1$ is frozen. The electromagnetic operator content of the six-vertex model is constituted by integer electric and half integer magnetic charges [28]. Depending on the parity of the lattice considered, the torus partition function reads

$$Z_c(g,f)=\frac{1}{\eta\bar\eta}\sum_{\substack{e\in\mathbb{Z}/f\\ m\in\mathbb{Z}f}}q^{(1/4)((e/\sqrt{g})+m\sqrt{g})^2}\bar q^{(1/4)((e/\sqrt{g})-m\sqrt{g})^2} \tag{9}$$

($f=1$ or $\frac{1}{2}$). With normalisations (8), the propagator reads $\langle\varphi(z,\bar z)\varphi(w,\bar w)\rangle=-(1/2g)\ln|z-w|^2$ and the stress tensor $T(z)=-g{:}(\partial_z\varphi)^2{:}$.

As shown by the Coulomb gas mapping which leads to (9), a physical (scalar) perturbation is represented in the continuum limit by a purely electric or magnetic operator. Up to a duality transformation (that exchanges e and m) and a global shift of φ (that exchanges cos and sin) this gives $\mathscr{O}=\cos\alpha\varphi$, $\alpha\in\mathbb{Z}$ or $g\mathbb{Z}/2$, $h=\bar h=\alpha^2/4g$. Hence $\mathscr{A}(1)$ is the action of a sine-Gordon model

$$\mathscr{A}=\int\left(\frac{g}{\pi}\partial_z\varphi\partial_{\bar z}\varphi+\lambda\cos\alpha\varphi\right)\mathrm{d}^2z. \tag{10}$$

In the generic g irrational case, the Virasoro characters for $\mathcal{O}$ and the identity are $\chi_h = q^h/P(q)$ and $\chi_0 = (1-q)/P(q)$. The existence of conserved quantities in the identity block $[\Phi_{11}]$ for grade $n \geq 1$ is excluded by a counting argument. For instance at grade 3 one has

$$(T^2) = g^2{:}(\partial\varphi)^4{:} - g{:}\partial^3\varphi\partial\varphi{:}$$

where (T^2) is the renormalised square of T. The residue of the single pole in the expansion of (T^2) by $\mathcal{O}$ reads

$$\mathrm{i}\alpha\left(1+\frac{\alpha^2}{4g}\right){:}\partial^3\varphi\, \mathrm{e}^{\mathrm{i}\alpha\varphi}{:} - 3\alpha^2{:}\partial\varphi\partial^2\varphi\, \mathrm{e}^{\mathrm{i}\alpha\varphi}{:} - 2\mathrm{i}\alpha g{:}(\partial\varphi)^3\, \mathrm{e}^{\mathrm{i}\alpha\varphi}{:}. \tag{12}$$

This is a total derivative for $h = \alpha^2/4g = 1$ only. The set of conserved quantities associated to (10) is built by considering all fields of integer dimension, with generating function $\Sigma_{p\geq 0}\, \chi_{p^2} = 1/P(q)$. Using the counting argument, one finds Q_n at $n = 1, 3, 5, 7$ and $n \geq 8$.

The excitation spectrum of the sine-Gordon model (10) has been studied in detail [29, 11]. For $h = \alpha^2/4g \geq \frac{1}{2}$, the only particles are the soliton-antisoliton pair. Hence at zero momentum one observes a mass M twice degenerate, plus a continuum starting at $2M$:

$$h \geq \tfrac{1}{2} \qquad M \times 2,\ \text{continuum above } 2M. \tag{13}$$

If $h < \frac{1}{2}$, bound states appear at masses $M^{(i)}$

$$h < \tfrac{1}{2} \qquad M^{(i)} = 2M \sin\left(\frac{\pi}{2}\frac{h}{1-h}\, i\right) \qquad i = 1, 2, \ldots, \frac{1-h}{h}. \tag{14}$$

In the transfer matrix spectrum, masses M ($M^{(i)}$) appear as deformations of U(1) charged (neutral) states.

The 'thermal' perturbation transforms the six-vertex model into an eight-vertex model. It is associated with a purely magnetic operator with $m = 2$ in (9) or $\alpha = 2g$, $h = g$. In this case the transfer matrix spectrum was explicitly calculated in [30] and agrees with the above (13), (14). The relation between the eight-vertex and the sine-Gordon model was established in [31]. Note that the six- (eight)-vertex model could be mapped alternatively on a Thirring (massive) theory with the same results [32, 33]. At $\Delta = 1$, the anisotropy operator coupled to Δ has a weight $h = 1$. In the ordered region $\Delta \to -1^-$, one should thus observe simply the soliton-antisoliton pair at M. This is confirmed by the exact solution [30].

Another case at $c = 1$ is the Ashkin-Teller model [25] which renormalises along its critical line onto a $\mathbb{Z}_2$ orbifold Gaussian theory. The partition function reads (see [34])

$$\begin{aligned} Z_{\mathrm{AT}} &= \tfrac{1}{2}Z_c(g, 2) + Z_{0\frac{1}{2}} + Z_{\frac{1}{2}0} + Z_{\frac{1}{2}\frac{1}{2}} \\ &= \tfrac{1}{2}Z_c(g, 2) + Z_c(4) - \tfrac{1}{2}Z_c(1) \end{aligned} \tag{15}$$

where g depends on the four-spin coupling. Equation (15) is obtained by showing the equivalence [34] between the AT model and a combination of six-vertex model sectors of different boundary conditions (BC). In particular the Z_{ab} correspond to a free field (8) with $(-1)^a((-1)^b)$ BC in the space (time) direction; these terms are necessary to reproduce the spin operator which is represented by a twist field [35] in the continuum limit. At $g = 1$, (15) coincides with the four-state Potts model, at $g = \frac{3}{4}$ with the $\mathbb{Z}_4$ model [36]. At $g = \frac{1}{2}$ ($\Delta = 0$), (15) is the square of the Ising model. The thermal exponent [37] of the AT or any of the latter models is given by $x_{\text{therm}} = 1/2g$.

For $T \neq T_c$, the equivalence leading to (15) still holds between the off-critical AT model and a staggered six-vertex model [25], where the perturbation is described by the electric operator $\alpha = 1$. Hence we expect the values of masses observed in the AT transfer matrix spectrum to belong to the set (13), (14). However, due to peculiar combination of sectors in (15), degeneracies may be different. This requires a more detailed analysis.

We first discuss the case $g = \frac{1}{2}$ (here the AT model still factorises on two independent Ising models). $h = \frac{1}{2}$, so the sine-Gordon mass spectrum is constituted by the soliton-antisoliton pair only. If $T \geq T_c$, a gap must appear between the ground state and the $a = \frac{1}{2}$ sector of (15); explicit analysis of [25] shows that the spin operators couple to the states at mass M, which is thus the value of this gap. The polarisation operator (product of two spins) correlations must decay with mass $2M$ since the two Ising models are decoupled. Other operators (like the energy) in the $a = 0$ sector must have correlations decaying at least as fast. Hence in this sector the masses M do not appear. This result is associated with the absence at $T = T_c$ of the electromagnetic operators of lowest dimension in the underlying vertex model ($m = \frac{1}{2}$, $h = g/16$) in $Z_c(g, 2)$. From this we conclude that the AT mass spectrum at $g = \frac{1}{2}$ for $T \to T_c^+$ is the same as the sine-Gordon one (13). For a single copy of the Ising model we thus find one particle at mass M plus the continuum above $2M$. This is confirmed by the exact solution [38].

If $T \leq T_c$, the AT model is ordered. In the thermodynamic limit, both Ising spins as well as the polarisation operator acquire a non-zero expectation value. For a finite system transfer matrix, the ground state, the first excited state of the $a = 0$ sector, and the two ground states of the $a = \frac{1}{2}$ sector will thus be asymptotically degenerate. From the discussion of [25, p 241], the spin operators do not couple to the states at mass M. Operators in the $a = 0$ sector, whose correlations decay faster than the spin ones, cannot couple to M either. Hence, above these four very close levels of the transfer matrix, the continuum starts directly. For a single copy of the Ising model, one should thus observe, for periodic BC, two asymptotically degenerate levels, plus a continuum. If a pure phase is selected by fixing BC, there is a single ground state before the threshold. Exact solution [38] confirms these predictions.

The asymmetry between high and low temperature spectra for the Ising model does not contradict duality, but is actually a consequence of it. Suppose we consider the $T \leq T_c$ transfer matrix spectrum and select a pure phase by choosing fixed BC with spins up say on both sides of the strip. Low temperature expansion [25] of the partition function is a sum over domain wall loops, with a factor $e^{-2/T}$ per link. Because of the BC, any path from one side of the strip to the other should intersect an even number of bonds. To obtain the mass spectrum we use the dual transform (which has the same partition function), and thus consider the loops as high temperature [25] $T' \geq T_c$ graphs, with $\tanh 1/T' = e^{-2/T}$. The fixed BC translate into free ones, but the constraint of an even number of crossed bonds excludes the mass M. Indeed, this mass is associated with the ground state of the $\mathbb{Z}_2$ odd sector, and such states propagate with an odd number of lines in the high temperature expansion. More generally, consider a self-dual model with isolated masses in the $T \to T_c^+$ region associated with the different order parameters. In the $T \to T_c^-$ region, these masses should be absent since, by duality, they are then associated with disorder parameters which cannot be observed in a pure phase.

Considerved quantities for the Ising model have been discussed at length. Using fields in $[\Phi_{11}]$ one finds they occur at any odd grade. Odd numbers can also be considered as A_1 exponents modulo $h^\vee = 2$.

Away from the decoupling point, it is reasonable to suppose that spin operators still couple to the states at mass M for $T \to T_c^+$, and that these states are absent from the $a=0$ sector. If $g > \frac{1}{2}$, neutral bound states appear in the sine-Gordon spectrum. Since for $T = T_c$, the scalar ($h = \bar{h}$) electromagnetically neutral operators of (15) contain all those of the six-vertex model, we expect these bounds states to be present in the $a=0$ sector of the AT model. Hence despite the sectors combination of (15), the mass spectrum of the AT model for $T \to T_c^+$ should be the same than the sine-Gordon one. For $T \to T_c^-$, since there are three order parameters, the duality [25] argument gives the mass spectrum (14) minus the three lower masses.

For $g = \frac{3}{4}$, we thus find in the $T \to T_c^+$ region particles at mass $M(\times 2)$, $\sqrt{2}M$ before the continuum. They correspond to the three sectors with eigenvalues $\omega, \omega^3, \omega^2$, ($\omega = e^{2i\pi/4}$) under $\mathbb{Z}_4$. This spectrum agrees with the S matrix constructed in [39], and the recent solution of [40]. If $T \to T_c^-$, the spectrum in a pure phase should be continuous. For the building of conserved quantities we must take into account the fields of (15) with h integer, $\bar{h} = 0$ (note in particular the current $h = 3$, $\bar{h} = 0$). Usual arguments establish the existence of conserved Q_n for the first exponents of A_3 modulo $h^{\upsilon} = 4$.

For $g = 1$ finally, (13), (14) gives particles at mass $M(\times 2)$, plus a bound state at mass M and $\sqrt{3}M$ before the continuum. The mass M degenerate three times is expected at $T \to T_c^+$ from S_4 symmetry of the four-state Potts model. It corresponds to the different order parameter sectors. The mass at $\sqrt{3}M$ should correspond to the ground state of energy like sector. If $T \to T_c^-$, the duality argument shows that the three masses M are not observed. The spectrum should thus contain the mass at $\sqrt{3}M$ only before the continuum.

Other off critical directions where $\mathcal{O}(1)$ is represented by an electric or magnetic operator can be studied in the same way. This is not the case of the $\mathbb{Z}_2$ symmetry breaking perturbation, since the spin operator is represented in the continuum limit by a twist field σ

$$\partial_z \varphi \sigma(w, \bar{w}) \sim (z-w)^{-1/2} \tau(w, \bar{w}).$$

This definition being scale invariant, the dimension of σ, $h = \bar{h} = \frac{1}{16}$, hence its two-point function, does not depend on the renormalised coupling g. This is not the case however for higher-order correlators, and we expect the mass spectrum of the AT model perturbed by a magnetic field $H(S_1 + S_2)$ to depend on g.

Additional information is known at the decoupling point $g = \frac{1}{2}$ from Zamolodchikov's work [7]. Since for a single Ising model the mass spectrum before the $2M$ threshold is

$$M, \qquad 2\sin\frac{9\pi}{30}M, \qquad 2\sin\frac{14\pi}{30}M \tag{16}$$

we have for the AT model at $g = \frac{1}{2}$

$$M(\times 2), \qquad 2\sin\frac{9\pi}{30}M(\times 2), \qquad 2\sin\frac{14\pi}{30}M(\times 2). \tag{17}$$

For $g \neq \frac{1}{2}$, this spectrum gets deformed as a function of g. We notice that blind application of the sine-Gordon spectrum for $h = \frac{1}{16}$ gives bound states at masses

$$M^{(i)} = 2\sin\frac{i\pi}{30}M \qquad i = 1, 2, \ldots, 15$$

a subset of which is (16) ($i = 9, 14$). The spectrum (16) cannot be obtained by considering perturbation by the polarisation operator at the decoupling point, as was tried earlier by Lüther [31].

3. $c<1$ models

We now turn to the unitary series with central charges [41]

$$c=1-\frac{6}{\mu(\mu+1)}. \tag{18}$$

We mainly deal with the diagonal series (AA modular invariants [42]) corresponding to $\mu-1$ multicritical models [43], or integrable IRF restricted models defined on A_μ diagrams [22]. These are still related to the six-vertex model, in presence however of some floating charges [21]. This is analogous to the construction of $c<1$ Virasoro algebra representations using the free bosonic field with charge at infinity [15, 16]. Due to the breaking of electrical neutrality it is not possible to use straightforwardly the sine-Gordon model (10) to obtain the mass spectrum when a relevant perturbation is turned on. A special situation occurs however in the case of Φ_{13} [10]

$$h_{13}=\frac{\mu-1}{\mu+1}. \tag{19}$$

We first discuss it from the lattice point of view.

At criticality, it has been shown how the A_μ transfer matrix spectrum can be recovered by combining μ sectors of the six-vertex model with twisted boundary conditions [23, 44]. The ground state of the rth sector defines the order parameter dimension h_{rr} associated with $\alpha_r=(r-1)\alpha_0$ (see below). A characterisation in terms of quantum A_1 symmetry is known for levels to be kept [23]. In particular the ground states of $r=\mu$ sector has to be excluded since $h_{\mu\mu}$ does not belong to the operator content [42]. The first excited state of each sector, associated with α_r-1, has also to be excluded. Combining all relevant levels gives an expression for A_μ partition function in terms of Gaussian ones [45] similar to (15)

$$Z_{A_\mu}=Z_c\left(\frac{\mu}{\mu+1},\mu+1\right)-Z_c\left(\frac{\mu}{\mu+1},1\right). \tag{20}$$

A lattice integrable extension of A_μ models is known [22], with elliptic Boltzmann weights, corresponding to the Φ_{13} direction. It is related this time to the eight-vertex model [25]. There is no such detailed information on the matching of levels, as in the critical case (20). But functional equations satisfied by transfer matrices indicate [22] that eigenvalues of the A_μ model are still included in the set of eight-vertex model eigenvalues obtained by combining μ sectors of different twisted boundary conditions (z term in (1.3.10) of [22]). Hence we reach again the conclusion that the mass spectrum of the A_μ model in the Φ_{13} direction should be the one of the eight-vertex (i.e. sine-Gordon) model, with possibility of different multiplicities. Note however that mass formulae (13), (14), due to breaking of electrical neutrality, have to be used not with h_{13} (20), but with the dimension of the magnetic ($m=2$) operator in the six-vertex model

$$h=g=\frac{\mu}{\mu+1}. \tag{21}$$

In the unitary series (19), $\mu\geqslant 3$. Hence $h>\frac{1}{2}$ in (21), and the sine-Gordon spectrum contains only the soliton-antisoliton pair at mass M. For restricted A_μ models, the alternative is thus simply between observing one mass M or not before the threshold.

We now recall that the critical point separates the regimes 3 and 4 [22]. Regime 3 is ordered for $\mu>2$, with $\mu-1$ phases associated with the $\mu-1$ order parameters (or the $\mu-1$ kept ground states of the μ sectors discussed above). If $\mu=3$, it corresponds to the low temperature region of the Ising model. Regime 4 is ordered for $\mu>3$, with $\mu-2$ phases. If $\mu=3$, it corresponds to the high temperature region of the Ising model.

If $\mu=3$, the mass spectrum in regime 3 is continuous, as discussed earlier. On the other hand we can also consider the $\mu\to\infty$ limit of A_μ models. It is known that scaling the distance to criticality t in such a way that t/μ remains fixed [46], one obtains weights of the six-vertex model perturbed in the $\Delta<-1$ direction. (In this process, the A_μ models are now considered at finite distance of criticality, but their spectrum being related to the eight-vertex model should be the same as for $T\to T_c$, hence allowing the argument about presence or not of a given mass). For the latter the spectrum contains the soliton antisoliton pair only. If $t=0$ however, the A_μ partition function goes to the periodic free field one

$$Z_{A_\mu}\sim\frac{\mu}{\sqrt{\operatorname{Im}\tau}\,\eta\bar{\eta}}\qquad \mu\to\infty. \tag{22}$$

Since in (22) all U(1) charged states are excluded, the soliton-antisoliton pair should not be observed. Hence in the limit $\mu\to\infty$, the A_μ mass spectrum should also be continuous. It is reasonable to suppose this holds true for any μ between 3 and ∞ as well.

In regime 4 case, the spectrum for $\mu=3$ contains a particle at mass M before the continuum. It is associated to a gap between the $r=1$ and $r=2$ sectors of the vertex model. If $\mu>3$, since there are only $\mu-2$ phases, the ground states of the $\mu-1$ sectors which were kept at criticality cannot all become degenerate. Inside a given sector, by analogy with what happens in regime 3 case, the spectrum should be continuous, starting at $2M$. But it may be that, as in the Ising case, the gap between $r=1$ and some r sector gives the isolated mass M, as is the case for $\mu=3$. In fact, numerical calculations show this is not the case for $\mu=4,5,6$, and hence likely for any μ. We thus reach the conclusion that the mass spectrum for A_μ models is continuous in all ordered regimes.

Hence diagonal unitary series perturbed by Φ_{13} lead to a continuous mass spectrum, except for the special case of the Ising model and $T\to T_c^+$ (regime 4) where there is a mass M before the threshold at $2M$. The case of non-unitary minimal models is more difficult because the sine-Gordon spectrum presents there additional bound states. For instance in the Lee-Yang case [47] $c=-\frac{22}{5}$, $g=\frac{2}{5}$, there is the soliton-antisoliton pair at M plus a bound state at $\sqrt{3}M$. In this simple situation, on the basis of $\mathrm{i}\varphi^3$ Lagrangian formalism [48], one expects however a single particle associated with φ to be present, hence observation in the minimal model of a single mass M before the continuum. This is confirmed by [8].

We now comment on some peculiarities of Φ_{13} form the point of view of conformal theory [10]. Recall that in the Coulomb gas approach, the dimensions are obtained by considering vertex operators of charge α and $2\alpha_0-\alpha$ as conjugate

$$h=\frac{\alpha^2-2\alpha\alpha_0}{4g} \tag{23}$$

where

$$g=\frac{\mu}{\mu+1} \qquad \alpha_0=\frac{1}{\mu+1} \tag{24}$$

($\gamma=1/(\mu+1)$ in (8)). There are two screening charges

$$\alpha_+=2 \qquad \alpha_-=-2\frac{\mu}{\mu+1} \tag{25}$$

and conformal weights in the Kac table are associated with

$$\alpha_{rs}=(1-r)\frac{\alpha_+}{2}+(1-s)\frac{\alpha_-}{2} \qquad 1\leqslant r\leqslant\mu-1 \qquad 1\leqslant s\leqslant\mu. \tag{26}$$

Hence h_{13} is associated with $\alpha_{13}=-\alpha_-=2\mu/(\mu+1)$, the opposite of α_- screening charge (note that h_{31} associated with $\alpha_{31}=-\alpha_+=-2$ corresponds to an irrelevant perturbation). This can prove useful in building conserved quantities. Indeed, on the basis of the counting argument, one finds for theories (19) perturbed by Φ_{13} and $\mu\geqslant5$ conserved quantities at grades $n=1,3,5,7$ only (for $\mu=3$, it goes up to $n=29$, $\mu=4$ up to $n=11$). However let us discuss in some detail $n=3$. One has

$$T(z)=-g{:}(\partial\varphi)^2{:}+\mathrm{i}\alpha_0\partial^2\varphi \tag{27}$$

hence the regularised square

$$(T^2)=-g{:}\partial^3\varphi\partial\varphi{:}+g^2{:}(\partial\varphi)^4{:}-\alpha_0^2{:}(\partial^2\varphi)^2{:}-2g\mathrm{i}\alpha_0{:}\partial^2\varphi(\partial\varphi)^2{:}+\frac{\mathrm{i}\alpha_0}{2}\partial^3\varphi. \tag{28}$$

When calculating short distance expansion with a vertex operator $\mathrm{e}^{\mathrm{i}\alpha\varphi}$, the last two terms which are total derivatives do not contribute to the single pole. We find the same result as (12), plus $-\mathrm{i}\alpha\alpha_0^2/g{:}\partial^3\,\mathrm{e}^{\mathrm{i}\alpha\varphi}{:}$. The condition for the residue of the single pole to be a total derivative reads

$$\alpha^4+2\alpha^2(2\alpha_0^2+4g)+16g^2=0 \tag{29}$$

solutions of which are

$$\alpha=\pm\alpha_\pm. \tag{30}$$

More generally, we know that screening operators commute with the Virasoro algebra, so the expansion of any polynomial in T and its derivatives with $\mathrm{e}^{\mathrm{i}\alpha_-\varphi}$ gives a single pole which is a total derivative. For polynomials which are even in $\varphi\to-\varphi$ (up to total derivatives like in T or (T^2)), this will also be the case for $\mathrm{e}^{-\mathrm{i}\alpha_-\varphi}$, hence for the physical perturbation $\mathcal{O}=\cos\alpha_-\varphi$. It should be easier to construct such polynomials than to work with the whole Virasoro system as in [7]. As an example, the conserved quantity at grade $n=9$, which is not given by the counting argument, is obtained in this way in [10]. There are several reasons [49] to suppose that such quantities occur for any odd n. Note that conserved quantities built in the Feigin-Fuchs framework may well be not observed in the minimal models themselves due to restriction of the Hilbert space. For instance in the Lee-Yang case, only a subset [8] (n not divisible by 3) of the odd numbers arises. For unitary models all odd numbers are expected.

We notice finally that the correct weight to be put in sine-Gordon spectrum (21) is obtained from formula (23) by removing the α_0 dependent part: $h=(-\alpha_-)^2/4g=g$.

4. Coset models

We turn now to A_n theories. We denote by $\boldsymbol{\Lambda}_\mu$, $\mu=1,\ldots,n$, the weights vectors, $\boldsymbol{r}_\mu$ the root vectors, with normalisations

$$\begin{aligned} \boldsymbol{r}_\mu \boldsymbol{r}_\nu &= 2\delta_{\mu\nu}-\delta_{\mu,\nu+1} \\ \boldsymbol{\Lambda}_\mu \boldsymbol{\alpha}_\nu &= \delta_{\mu\nu}. \end{aligned} \tag{31}$$

An integrable $(n+1)(2n+1)$-vertex model [50] generalising the six-vertex model is known, whose bond variables are the set of $n+1$ vectors

$$\boldsymbol{\mu}=\boldsymbol{\Lambda}_{\mu+1}-\boldsymbol{\Lambda}_\mu \qquad \mu=0,\ldots,n \tag{32}$$

with conventions $\boldsymbol{\Lambda}_0=\boldsymbol{\Lambda}_{n+1}=0$. Dual face variables are heights of a solid-on-solid model defined on the weight lattice dilated by a factor $\sqrt{2}\pi$ (this is necessary to respect the familiar conventions of (8) and (31)). This model has a critical line parametrised by a parameter $\gamma\in[0,\pi]$, and renormalises onto the free n component bosonic theory with $c=n$ and [51, 52]

$$\mathcal{A}^*=\frac{g}{\pi}\int \partial_z\boldsymbol{\varphi}\partial_{\bar z}\boldsymbol{\varphi}\ \mathrm{d}^2z \tag{33}$$

where $g=1-(\gamma/\pi)$ [53]. The propagator of (33) is

$$\langle\boldsymbol{\varphi}(z,\bar z)\boldsymbol{\varphi}(w,\bar w)\rangle=-(n/2g)\ln|z-w|^2.$$

As in A_1 case, a non-critical extension—Belavine model—of the above $(n+1)\times(2n+1)$ vertex model has been written where additional vertices are introduced, and Boltzmann weights are given by elliptic functions [50]. The associated exponent [55] is still (21).

Most interesting systems related to A_n have $c<n$. In the conformal framework, they are obtained by a generalised [17-19, 51] Feigin-Fuchs construction based on the stress tensor

$$T=-g{:}(\partial\boldsymbol{\varphi})^2{:}+\mathrm{i}\alpha_0\boldsymbol{\rho}\partial^2\boldsymbol{\varphi} \tag{34}$$

where

$$\boldsymbol{\rho}=\boldsymbol{\Lambda}_1+\boldsymbol{\Lambda}_2+\ldots\boldsymbol{\Lambda}_n \qquad \boldsymbol{\rho}^2=\frac{n(n+1)(n+2)}{12}.$$

Writing $g=\mu/(\mu+1)$, one has $\alpha_0=1/(\mu+1)$ and

$$c=n-\frac{12\alpha_0^2\boldsymbol{\rho}^2}{g}=n\left[1-\frac{(n+1)(n+2)}{\mu(\mu+1)}\right]. \tag{35}$$

For comparison with (7) one has

$$h^{\vee}=n+1 \qquad k=\mu-n-1 \tag{36}$$

Vertex operators of charge $\boldsymbol{\alpha}$ and $(2\sqrt{2}\alpha_0\boldsymbol{\rho}-\boldsymbol{\alpha})$ are conjugate, with weight

$$h=\frac{1}{4g}(\boldsymbol{\alpha}^2-2\sqrt{2}\alpha_0\boldsymbol{\rho}\cdot\boldsymbol{\alpha}). \tag{37}$$

There are $2n$ screening operators of charges

$$\boldsymbol{\alpha}=\sqrt{2}\alpha_{\pm}\boldsymbol{r}_\mu \tag{38}$$

with $\alpha_+=2$, $\alpha_-=-2\mu/(\mu+1)$ as in (25).

From a lattice point of view, the connection between the $(n+1)(2n+1)$ vertex model and the restricted IRF models is not as fully understood as for $n=1$. For the diagonal unitary series however, i.e. models related to diagonal modular invariants [42], whose configuration space is [54] the set of dominant integral weights of level $k+1=\mu-n$, to which we restrict here, it involves also [51] a combination of sectors of the vertex model with different boundary conditions. Expressions similar to (20) have been written [51].

The non-critical elliptic extension of restricted models has been studied in [54]. Two regimes (3 and 4) can be reached which are ordered except for the case $k=1$ and regime 4, with associated exponent

$$h=\frac{\mu-n}{\mu+1}=\frac{k+1}{k+n+2}. \tag{39}$$

Functional relations satisfied by eigenvalues do not seem to have been studied so far. Intertwining vectors are however known, which indicate that the mass spectrum of restricted models perturbed by operator (39) should be the same, with the possibility of different multiplicities, than that of the Belavin model, i.e. of the $(n+1)(2n+1)$ vertex model perturbed by (21). The electric charge associated to (39) is related to the highest root of A_n

$$\boldsymbol{\alpha}=-\sqrt{2}\alpha_-\sum_{\mu=1}^{n}\boldsymbol{r}_\mu. \tag{40}$$

Since the Belavin model exhibits $\mathbb{Z}_{n+1}$ symmetry, we must thus write its continuum limit action (1) with an operator $\mathcal{O}$ that contains all $\mathbb{Z}_{n+1}$ transforms of (40). These turn out to be the $n\,\alpha_-$ like screening charges $\boldsymbol{\alpha}=\sqrt{2}\alpha_-\boldsymbol{r}_\mu$. Hence, the perturbation being, moreover, real,

$$\mathscr{A}=\int\left\{\frac{g}{\pi}\partial_z\boldsymbol{\varphi}\partial_{\bar z}\boldsymbol{\varphi}+\lambda\left[\sum_{\mu=1}^{n}\exp(\mathrm{i}\sqrt{2}\alpha_-\boldsymbol{r}_\mu\boldsymbol{\varphi})+\exp\left(-\mathrm{i}\sqrt{2}\alpha_-\sum_{\mu=1}^{n}\boldsymbol{r}_\mu\boldsymbol{\varphi}\right)+\mathrm{cc}\right]\right\}\mathrm{d}^2z \tag{41}$$

(of course all vertex operators in $\mathcal{O}$ have the same dimension (21) in the free theory (33)). This is essentially the action of a Toda field theory [26, 56]. The latter however should not contain the CC term of (41). It can be suppressed by looking instead at equations of motion for the holomorphic (at $\lambda=0$) component of the field $\boldsymbol{\varphi}_R$, one finds

$$\partial_z\partial_{\bar z}\boldsymbol{\varphi}_R=-\pi\frac{\lambda}{2g}\frac{\delta}{\delta\boldsymbol{\varphi}_R}\left(\sum_{\mu=1}^{n}\exp(\mathrm{i}\sqrt{2}\alpha_i\boldsymbol{r}_\mu\boldsymbol{\varphi}_R)+\exp\left(-\mathrm{i}\sqrt{2}\alpha_-\sum_{\mu=1}^{n}\boldsymbol{r}_\mu\boldsymbol{\varphi}_R\right)\right) \tag{42}$$

which is Toda system based on $A_n^{(1)}$ [57].

We now discuss the mass spectrum of (42). We note first that the $(n+1)(2n+1)$ vertex model exhibits $\mathrm{SU}(n+1)\times\mathrm{SU}(n+1)$ symmetry at $\gamma=0$. Instead of the non-critical elliptic Belavin model, one can consider there the extension keeping the same set of vertices, but with γ purely imaginary (and hence hyperbolic Boltzmann weights). This corresponds to perturbing (33) with a marginal operator $\alpha_- = -2$ in (41). Then the spectrum is known to be independent of γ up to a global scale [52], and is given by the spectrum of the nonlinear sigma model [27, 58] on the group $SU(n+1)$. Fundamental particles have masses

$$M^{(i)} = \sin\frac{i\pi}{n+1} \Big/ \sin\frac{\pi}{n+1} (\times 2) \qquad i = 1, \ldots, n. \tag{43}$$

In the lack of any further information, we make the hypothesis that, as in the sine-Gordon case, spectrum (43) is still observed when one perturbs (33) as in (41) with $\alpha_- < -2$. It should correspond to the charged solitons of the theory. Moreover, as in (14), for coupling $1-h$ small enough, no additional bound states should appear. In the following we suppose this holds true at least for $h \geqslant (n+2)/(n+3)$. With these hypotheses, the spectrum of restricted models perturbed by (39) is given, up to different multiplicities, by

$$M^{(i)} = M^{(n+1-i)} = \sin\frac{i\pi}{n+1} \Big/ \sin\frac{\pi}{n+1} \qquad i = 1, \ldots, n. \tag{44}$$

An interesting case occurs for $k=1$ where the restricted model is identical to the parafermion $\mathbb{Z}_{n+1}$ model of [36]. For $n=2$ it corresponds in particular to the three-state Potts model, for $n=3$ to the $\mathbb{Z}_4$ model already discussed. In the disordered regime 4, the mass spectrum should be exactly (44), masses $M^{(i)}$ corresponding to the different order parameters. The direction (39) corresponds to the usual thermal perturbation, preserving in particular $\mathbb{Z}_{n+1}$ symmetry. This prediction agrees with the results of [39, 40], which seems to confirm the validity of our assumptions. Regime 3 is ordered, and by the duality argument [60], the spectrum there should be continuous. In $k \to \infty$ limit, the same argument as in (22) can be used to show that the spectrum in regime 3 is still continuous. Probably this holds true for finite k as well. In regime 4 we do not have arguments or numerical calculations to determine what happens if $k > 1$. By analogy with $n=1$ case, we suspect the spectrum to be as well continuous.

We notice that $\mathbb{Z}_{n+1}$ transforms of charge $\boldsymbol{\alpha}$ in (40) are the screening charges (38), as was the case for $n=1$. For $n>1$, the symmetry algebra is generated [19] by the stress tensor T and $n-1$ higher spin operators W associated with higher-order Casimirs of A_n. The screening operators commute with T and all the W. Hence conserved quantities are obtained by building polynomials in T, W and their derivatives which are $\mathbb{Z}_{n+1}$ symmetric (up to total derivatives). One expects [10] by analogy with the classical case [26] that the corresponding grades are given by the exponents of A_n modulo the Coxeter $h^v = n+1$. This can be established for the first few numbers.

We now consider D and E algebras. In the D case, the underlying critical vertex model is known [20]; in the E case, weights have been obtained at the higher symmetry $(G \times G)$ point only [27]. Not much is known on the possible restricted IRF models. Some information can however be obtained from free field construction of related conformal theories.

One starts there with a r component bosonic field and a modified stress tensor as in (34). Here

$$\boldsymbol{\rho}^2 = r\frac{h^v(h^v+1)}{12}. \tag{45}$$

With g and α_0 as before one has

$$c=r\left[1-\frac{h^{v}(h^{v}+1)}{12}\right]. \tag{46}$$

For comparison with (7)

$$k=\mu-h^{v}. \tag{47}$$

Hence one finds for D_n algebras

$$c=n\left[1-\frac{(2n-2)(2n-1)}{(2n-2+k)(2n-1+k)}\right] \qquad c=1 \text{ for } k=1 \tag{48}$$

and for the exceptional cases

$$\begin{aligned}
&\mathrm{E}_6: \quad && c=6\left[1-\frac{166}{(12+k)(13+k)}\right] \qquad && c=\tfrac{6}{7} \text{ for } k=1\\
&\mathrm{E}_7: \quad && c=7\left[1-\frac{342}{(18+k)(19+k)}\right] \qquad && c=\tfrac{7}{10} \text{ for } k=1\\
&\mathrm{E}_8: \quad && c=8\left[1-\frac{930}{(30+k)(31+k)}\right] \qquad && c=\tfrac{1}{2} \text{ for } k=1.
\end{aligned} \tag{49}$$

For D_n at $k=1$ the central charge is unity. The corresponding models should be points on the Gaussian (six-vertex) or orbifold (Ashkin–Teller) lines. For E_8 at $k=1$ we find the central charge of the Ising model, for E_7 at $k=1$, the one of the tricritical Ising model. In both cases there is only one modular invariant [42], and one checks that all the corresponding characters can be obtained in the coset construction. For E_6 at $k=1$ we get the central charge of the 5th critical Ising or tricritical three-state Potts model. However $\mathrm{E}_6^{(1)}$ has only nine representations at level 2; hence only the tricritical three-state Potts model can be considered as arising from the coset.

Screening operators are built as in (38), where $\boldsymbol{r}_\mu$ are the simple roots.

It does not seem that any elliptic-like vertex models associated with D or E has been written, possibly for some fundamental reason [61]. We may suppose however that integrable off-critical directions exist, because in particular of the existence of a (classically) integrable Toda system [56]. The latter looks as (42) where the set of simple roots has to be completed by $(-\sqrt{2}\alpha_-)\times$ the highest root. The associated exponent for the vertex model should still be given by (21). For the restricted model, we find using (37)

$$h=\frac{\mu-h^{v}+1}{\mu+1}=\frac{k+1}{k+h^{v}+1}. \tag{50}$$

Conserved quantities are expected to occur at grades given by the exponent of $\mathscr{G}$ modulo h^{v}.

As in the A_n case, the mass spectrum for D or E vertex models in the hyperbolic weights region, hence of the Toda field theory with $h=1$ perturbing operator, is the one of the nonlinear sigma model with $\mathrm{G}\times\mathrm{G}$ symmetry [52]. We conjecture again this holds true for $h<1$, and that at least for $h\geqslant(h^{v}+1)/(h^{v}+2)$, no bound state is formed. Hence the mass spectrum of models (48) or (49) perturbed in direction (50) should be the one calculated in [27], with maybe different degeneracies.

In the D_n sigma model, fundamental particles have masses

$$M^{(i)} = 2M \sin \frac{i\pi}{2(n-1)} \qquad i = 1, 2, \ldots, n-2 \qquad M^{(n-1)} = M^{(n)} = M. \tag{51}$$

For $k = 1$ the coset models have $c = 1$, while the dimension of the perturbating field is $h = 1/n$. In this case (51) is also the spectrum which would be obtained by using the sine-Gordon mass formulae (13), (14), as expected. This constitutes an interesting cross check of our hypothesis.

Algebra E_8 at $k = 1$ corresponds to the Ising model in a magnetic field. There indeed the spectrum of [27] is the same than the one obtained by Zamolodchikov. Since there is no broken symmetry phase, it is observed on both sides of the critical point. This was confirmed numerically [9].

Algebra E_7 at $k = 1$ corresponds to the tricritical Ising model [62] perturbed by the energy operator, $h = \frac{1}{10}$. The masses of elementary particles for the sigma model are

$$\begin{gathered} M^{(1)} = M \qquad M^{(2)} = 2 \sin \frac{2\pi}{9} M \qquad M^{(3)} = 2 \cos \frac{\pi}{9} M \qquad M^{(4)} = 2 \cos \frac{\pi}{18} M \\ M^{(5)} = 4 \cos \frac{\pi}{18} \sin \frac{2\pi}{9} M \qquad M^{(6)} = M/2 \sin \frac{\pi}{18} \qquad M^{(7)} = 4 \cos \frac{\pi}{18} \cos \frac{\pi}{9} M. \end{gathered} \tag{52}$$

Only $M^{(1)}, M^{(2)}, M^{(3)}, M^{(4)}$ are under the $2M$ threshold. For an E_7 coset model we thus expect the mass spectrum to be given (up to different degeneracies) by

$$M, \qquad 2 \sin \frac{2\pi}{9} M, \qquad 2 \cos \frac{\pi}{9} M, \qquad 2 \cos \frac{\pi}{18} M. \tag{53}$$

In the Ising tricritical case ($k = 1$) and for $T \to T_c^+$, (53) is indeed observed numerically (see below). Regarding conserved quantities, the counting argument shows some occur for $n = 1, 5, 7, 9, 11, 13, 17$ which are E_7 exponents $h^v = 18$. As usual one supposes they can be built for other n by moding out with h^v. For $T \to T_c^-$, one finds numerically a continuous spectrum.

Algebra E_6 at $k = 1$ corresponds to the tricritical three state Potts model perturbed by the thermal operator $h = \frac{1}{7}$. The fundamental particles have here masses

$$M^{(1)} = M(\times 2) \qquad M^{(2)} = \sqrt{2} M \qquad M^{(3)} = \frac{\sqrt{3}+1}{\sqrt{2}} M(\times 2) \qquad M^{(4)} = (\sqrt{3}+1) M. \tag{54}$$

$M^{(1)}, M^{(2)}, M^{(3)}$ are under the $2M$ threshold and should be observed for $T \to T_c^+$. States associated with $M^{(1)}, M^{(3)}$ have non-zero $\mathbb{Z}_3$ charge. Corresponding masses are probably not observed when $T \to T_c^-$, although we do not know a self-dual version of the tricritical three-state Potts model to justify it. Regarding conserved quantities, the counting argument gives grades $n = 1, 5, 7, 11$ in the identity block. Additional ones at grades $n = 4, 8$ can be obtained using the block $[\Phi_{51}]$, where $h_{51} = 5$. This would not be possible in the 5th critical Ising model where there is no field of weights $(5, 0)$). Hence we find the exponents of E_6.

5. Numerical checks

Our predictions relying on some plausible but not rigorously justified steps, we feel it worthwhile to carry out a few numerical checks. The general technique consists in studying transfer matrices structure for models close to criticality, where the level gaps are expected to fit with mass spectra discussed. To reproduce our continuum limit results in lattice observations one should have $L \gg \xi = M^{-1} \simeq \lambda\, 1/[2(h-1)] \gg a$ where L is the finite size system (strip width) and a is the lattice spacing. In numerical calculations, one can check that this scaling regime is reached by plotting [9] the various measured masses (of order i) as functions $f^{(i)}$ $(z = L\lambda\, 1/[2(1-h)])$: these must lie on a single curve as L and λ vary. Predicted mass ratios are observed in the $z \to \infty$ limit. The critical dimensions are on the other hand obtained by considering the finite size scaling behaviour of gaps at $\lambda = 0$, which corresponds to the opposite limit $z \to 0$ of functions $f^{(i)}$.

We start by considering the Ashkin–Teller model perturbed by the thermal operator. For technical convenience, calculations have been carried out using the Hamiltonian version of [63]. In table 1 we give a few results for the measured mass spectrum for $T \to T_c^+$ at several points on the critical line. ε is the four-spin coupling, related to the renormalised constant in (8) by $g = 1/\pi \cos^{-1}(-\varepsilon)$. The mass M (twice degenerate) is set equal to 1. Between the Ising decoupling point and the $\mathbb{Z}_2$ point, finite size corrections are described by a power law and the convergence of conventional extrapolation algorithms is comparable with that found in the Ising model [9]. Between the $\mathbb{Z}_4$ and the Potts point however, the leading correction becomes logarithmic. The convergence of the data becomes difficult when approaching the four-state Potts model point, and final numbers obtained by a double extrapolation procedure [9] have rather large error bars. Nevertheless our predictions are confirmed, especially the appearance of a fourth mass between the $\mathbb{Z}_4$ $(\varepsilon = \sqrt{2}/2)$ and the Potts points. In the case $T \to T_c^-$

Table 1. Numerical results for the third and fourth masses of the Ashkin–Teller model and $T \to T_c^+$. ε is the four spins coupling. $M^{(1)} = M^{(2)} = 1$.

ε	0.25	$1/\sqrt{2}$	0.875	0.92	1.0
$M^{(3)}$	1.853 (8)	1.40 (5)	1.24 (5)	1.17 (5)	1
$M^{(4)}$	2.0 (1)	2.0 (1)	1.94 (5)	1.91 (5)	1.8 (2)

Table 2. Predictions for the mass spectrum of several coset models perturbed by the operator of weight h.

A_μ IRF models $\mu \geq 4$		Continuous in regime 3 and 4	$h = \dfrac{\mu-1}{\mu+1}$	(ϕ_{13})
Z_{n+1} models	$T \to T_c^+$	$M^{(i)} = M^{(n+1-i)} = \sin \dfrac{i\pi}{n+1} \Big/ \sin \dfrac{\pi}{n+1}$, $i = 1, \ldots, n$	$h = \dfrac{2}{n+3}$	(thermal)
	$T \to T_c^-$	Continuous		
Ising model (E_8)	$H \to 0^{\pm}$	$M,\ 2\cos\dfrac{\pi}{5} M,\ 2\cos\dfrac{\pi}{30} M$	$h = \frac{1}{16}$	(magnetic)
Tricritical Ising (E_7)	$T \to T_c^+$	$M,\ 2\sin\dfrac{2\pi}{9} M,\ 2\cos\dfrac{\pi}{9} M,\ 2\cos\dfrac{\pi}{18} M$	$h = \frac{1}{10}$ (thermal)	
	$T \to T_c^-$	Continuous		
Tricritical three-state Potts model	$T \to T_c^+$	$M(\times 2), \sqrt{2}M, \dfrac{\sqrt{3}+1}{\sqrt{2}} M(\times 2)$	$h = \frac{1}{7}$	(thermal)

our results agree with a continuum spectrum up to the $\mathbb{Z}_4$ point. Around the four-state Potts point, the convergence is not good enough to give information on the fourth mass.

Another test of interest concerns the models (18) perturbed by the Φ_{13} ($h = \frac{3}{5}$) operator. There calculations have been carried out using the IRF models of [22], and Boltzmann weights given by elliptic functions of nome q. In this case, the predicted spectrum being related to the eight-vertex model should be observed at finite distance of the critical point as well [30], which makes a numerical test easier. We indeed observed with great accuracy results confirming the continuum in regimes 3 and 4 for A_μ models and $\mu = 4, 5, 6$. For instance in the tricritical Ising model case (here Φ_{13} corresponds to the vacancy operator) we give estimates of the five first masses in regime 4 as $q \to 0$

$$\begin{array}{lll} M^{(1)} = 1 & M^{(2)} = 1.00(5) & M^{(3)} = 1.1(1) \\ M^{(4)} = 1.1(1) & M^{(5)} = 1.2(1). & \end{array} \tag{55}$$

Finally we give results for the tricritical Ising model perturbed in the thermal direction ($h = \frac{1}{10}$). Calculations have been carried out on the A_4 model [22] by adding to the critical Boltzmann weights a perturbation proportional to the lattice realisation of Φ_{12} [64]. If $T \to T_c^+$, the estimates of the four first masses are

$$M^{(1)} = 1 \qquad M^{(2)} = 1.26(3) \qquad M^{(3)} = 1.88(5) \qquad M^{(4)} = 1.95(5) \tag{56}$$

in agreement with the E_7 prediction (53).

6. Conclusion

To conclude, it is clear that we have just realised a few steps in the program proposed to analyse coset models off criticality. Nevertheless, the circumstances are fortunate enough that a few assumptions allowed us, using already known results, to derive the mass spectrum of several interesting models. Actually these spectra appear to enjoy more universality than the critical properties. In particular it is striking that with one possible exception (Ashkin–Teller model for $g \geqslant \frac{3}{4}$) the mass spectrum is continuous in all ordered phases encountered [65]. Among things which remain to be understood are the proper derivation of the mass spectrum for Toda theories, and the restriction procedure that relates them to coset models off criticality. On the lattice, this latter question may involve using the elliptic extension of the quantum groups [66]. Also it would be interesting to derive the S matrices themselves from those written in [27].

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Note added in proof. Since the submission of this paper, several beautiful papers have solved most of the questions we raised. See, for instance, the recent preprints by Christe-Musseido, Braden *et al*, Smirnov and references therein.

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